Assignment 8.

This homework is due *Thursday*, October 23.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 5.

1. Quick reminder

Simple Approximation Lemma. Let $f : E \to \mathbb{R}$ be measurable. Assume f is bounded on E. Then for each $\varepsilon > 0$, there are simple functions φ_ε, ψ_ε defined on E such that

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon, \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } E.$$

Egoroff's Theorem. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\varepsilon > 0$, there is a closed set F contained in E for which

$$\{f_n\} \to f \text{ uniformly on } F, \text{ and } m(E \setminus F) < \varepsilon.$$

Lusin's Theorem. Let $f : E \to \mathbb{R}$ be measurable. Then for each $\varepsilon > 0$, there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a closed set F contained in E such that

$$f = g \text{ on } F, \text{ and } m(E \setminus F) < \varepsilon.$$

Lebesgue integral of bounded function. For a bounded function f on a set E of finite measure, Lebesgue integral $\int_E f$ is defined as the common value of $\sup\{\int_E \varphi \mid \varphi \leq f, \varphi \text{ simple}\}$ and $\inf\{\int_E \psi \mid \psi \geq f, \psi \text{ simple}\}$, if the latter two are equal (which is guaranteed if f is measurable).

2. Exercises

(1) (3.2.12) Let f be a bounded measurable function on E. Show that there are sequences of simple functions on E, $\{\varphi_n\}$ and $\{\psi_n\}$, such that $\{\varphi_n\}$ is increasing, $\{\psi_n\}$ is decreasing and each of these sequences converge to f uniformly on E. (*Hint:* Use Simple Approximation Lemma.)

(2) (3.2.21-) For a sequence $\{f_n\}$ of measurable functions with common domain E, show that $\inf\{f_n\}$ and $\sup\{f_n\}$ are measurable. (*Hint:* Express sup and inf as limits of appropriate sequences; or directly use the definition of a measurable function.)

(3) (3.3.26) Let $E = [0, 1]$. Lusin's theorem provides that a measurable $f : E \to \mathbb{R}$ equals to some continuous g at points of a set $F \subset E$. Must f itself be continuous at any point (as a function on E)?

(4) (3.3.27) Show that the conclusion of Egoroff's theorem is false if we drop the assumption that the domain E has finite measure.(*Hint:* Take a function f that is zero on $\mathbb{R}_-$ and nonzero on $\mathbb{R}_+$, consider $\{f(x - n)\}$.)

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(5) (3.3.31) Let $\{f_n\}$ be a sequence of measurable functions on E that converges to the real-valued f pointwise on E. Show that $E = E_0 \cup \bigcup_{k=1}^{\infty} E_k$, where all sets $E_0, E_1, \ldots$ are measurable, and $\{f_n\}$ converges uniformly to f on each E_k if $k > 0$, and $m(E_0) = 0$.

(6) (4.1.6+) Prove that a continuous function on a closed interval $[a, b]$ is Riemann integrable. (*Hint:* Use uniform continuity to put an upper bound on the difference between upper and lower Darboux sums.)

(7) (4.2.9) Let E have measure zero. Show that if f is a bounded function on E, then f is measurable and $\int_E f = 0$.

(8) Let f be a simple function on a set E of finite measure. Then two definition of Lebesgue integral apply to f: the Lebesgue integral of a simple function, and the Legesgue integral of a bounded function. Show that these definitions agree on f, i.e. they both give the same value of $\int_E f$.